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$$MV - MF = \text{constant.}$$

$$MV_1 - MF = \text{constant.}$$

Hence

$$MV - MV_1 = \text{constant.}$$

Q. E. D.

Property (D):

M, M' being any two fixed points on the ellipse (E), and V any point on the hyperbola (H),¹

$$VM - VM' = \text{constant.}$$

Proof:

$$MV + MF' = M'V + M'F' = \text{constant.}$$

Hence

$$VM - VM' = \text{constant.}$$

Q. E. D.

It can be proved without difficulty that the locus of the points having the property (B) is the hyperbola (H), as well as that the locus of the points having the property (C) is the ellipse (E).

II. A PROPERTY OF HOMOGENEOUS FUNCTIONS.

By J. E. TREVOR, Cornell University.

When a body constituted of two independent component substances and subject to no mechanical or thermal separation of its parts is in a state of thermodynamic equilibrium, the body may exhibit distinct liquid or aëriform parts. Any such part has variable mass, composition, and thermodynamic state, and is termed a fluid "phase" of the body. When x_1, x_2, x_3, x_4 denote the volume, the entropy, and the component-masses of a two-component phase, the energy of the phase is a homogeneous function $E(x_1, x_2, x_3, x_4)$ such that

$$(1) \quad t \cdot E(x_1, x_2, x_3, x_4) = E(tx_1, tx_2, tx_3, tx_4),$$

where t is any positive number.² The specific volume, specific entropy, and specific component-masses of the body are defined by the equations

$$y_1 = \frac{x_1}{x_3 + x_4}, \quad y_2 = \frac{x_2}{x_3 + x_4}, \quad y_3 = \frac{x_3}{x_3 + x_4}, \quad y_4 = \frac{x_4}{x_3 + x_4} = 1 - y_3.$$

When $t = 1/(x_3 + x_4)$, and X is written for $x_3 + x_4$, the equation (1) becomes

$$E = X \cdot E(y_1, y_2, y_3, 1 - y_3)$$

$$(2) \quad = X \cdot e(y_1, y_2, y_3),$$

where the "specific energy" e of the phase is a function of the three variables y_1, y_2, y_3 .

¹ Properties (B), (C) and (D) are due to Dupin, *Correspondance sur l'école polytechnique*, Jan., 1807, p. 218, and Jan., 1813, p. 424. For further references in this connection, and to generalizations by Chasles and Plücker, the *Encyclopédie des Sciences Mathématiques*, tome III, vol. 4, pp. 81-86, and tome III, vol. 3, pp. 51-52, may be consulted.—EDITOR-IN-CHIEF.

² Functions which possess this restricted form of homogeneity are termed "positively homogeneous" by Bolza, *Lectures on the Calculus of Variations*, Chicago, 1904, p. 119.—EDITOR.

If we write Taylor's expansion in the notation $\delta E = \delta^1 E + \delta^2 E + \dots$, where $\delta^n E$ is the sum of terms of the n th order, we have that the criterion of the stability of the thermodynamic equilibrium of a fluid phase is $\delta^2 E \geq 0$, where the sign of equality holds only when the variations of the variables satisfy the conditions

$$(3) \quad \frac{\delta x_1}{x_1} = \frac{\delta x_2}{x_2} = \frac{\delta x_3}{x_3} = \frac{\delta x_4}{x_4}.$$

In the desire to express the conditions of stability in terms of derivatives of the specific energy $e(y_1, y_2, y_3)$, it becomes necessary to establish a relation between the sum of terms $\delta^2 E$ and the variations of the function e . Giving the four variables x_i independent increments δx_i , the functions E, e in (2) obtain increments $\delta E, \delta e$, and we have

$$\delta E = e\delta X + X\delta e + \delta X\delta e,$$

or

$$(4) \quad \delta^1 E + \delta^2 E + \dots = e\delta X + X \cdot \delta^1 e + X \cdot \delta^2 e + \dots + \delta X \cdot \delta^1 e + \delta X \cdot \delta^2 e + \dots.$$

The first member of this equation is a power series in $\delta x_1, \delta x_2, \delta x_3, \delta x_4$, with constant coefficients. The second member is a power series in $\delta y_1, \delta y_2, \delta y_3, \delta X$, with constant coefficients. Now, since $x_i = Xy_i$, the variables δx_i have the values

$$(5) \quad \delta x_i = (X + \delta X)\delta y_i + y_i \delta X \quad (i = 1, 2, 3, 4),$$

where $y_4 = 1 - y_3$ and $\delta y_4 = -\delta y_3$. If we replace the δx_i in the first member of (4) by these values, both members of (4) will be power series in $\delta y_1, \delta y_2, \delta y_3, \delta X$.

The first term of the first member of (4) is a sum of terms $\delta^1 E = \sum A_i \delta x_i$, or by (5),

$$(6) \quad \delta^1 E = X \cdot \sum A_i \delta y_i + \delta X \cdot \sum A_i \delta y_i + \delta X \cdot \sum A_i y_i.$$

The first, and no other, of these three expressions contains δy_i with no factor δX . No terms in δy_i alone occur in $\delta^2 E, \delta^3 E, \dots$, since by (5) the terms occurring there are at least quadratic in the variables. So the first expression in the right-hand member of (6) is equal to the sum of the terms in δy_i in the second member of (4),

$$(7a) \quad X \cdot \sum A_i \delta y_i = X \cdot \delta^1 e.$$

It follows that $\delta^1 e = \sum A_i \cdot \delta y_i$; wherefore the second expression in (6) is

$$(7b) \quad \delta X \cdot \sum A_i \delta y_i = \delta X \cdot \delta^1 e.$$

The third expression in (6) is in δX . No terms in δX alone occur in $\delta^2 E, \delta^3 E, \dots$, since these terms are at least quadratic. So the third expression in (6) is equal to the term in δX in the second member of (4),

$$(7c) \quad \delta X \cdot \sum A_i y_i = e \cdot \delta X.$$

On substituting these three results, (6) becomes

$$(8) \quad \delta^1 E = (X + \delta X)\delta^1 e + e\delta X.$$

On subtracting (8) from (4), member by member, there remains

$$(9) \quad \delta^2E + \delta^3E + \cdots = X \cdot \delta^2e + X \cdot \delta^3e + \cdots + \delta X \cdot \delta^2e + \delta X \cdot \delta^3e + \cdots.$$

The first term of the first member of (9) is a sum of terms $\delta^2E = \Sigma A_{ij}\delta x_i\delta x_j$, or by (5)

$$\delta^2E = \Sigma A_{ij}[(X + \delta X)\delta y_i + y_i\delta X][(X + \delta X)\delta y_j + y_j\delta X].$$

Expanding the indicated product,

$$(10) \quad \begin{aligned} \delta^2E = & X^2 \cdot \Sigma A_{ij}\delta y_i\delta y_j + 2X\delta X \cdot \Sigma A_{ij}\delta y_i\delta y_j + (\delta X)^2 \cdot \Sigma A_{ij}\delta y_i\delta y_j \\ & + X\delta X \cdot \Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) + (\delta X)^2 \cdot \Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) \\ & + (\delta X)^2 \cdot \Sigma A_{ij}y_iy_j. \end{aligned}$$

The first, and no other, of these six expressions is in $\delta y_i\delta y_j$. No terms in this product alone occur in δ^3E , δ^4E , \cdots , since by (5) the terms occurring there are at least cubic in the variables. So the first expression in (10) is equal to the sum of terms in $\delta y_i\delta y_j$ in the second member of (9),

$$(11a) \quad X^2 \cdot \Sigma A_{ij}\delta y_i\delta y_j = X \cdot \delta^2e.$$

It follows that $\delta^2e = X \cdot \Sigma A_{ij}\delta y_i\delta y_j$; wherefore the second and third expressions in (10) are

$$(11b) \quad 2X\delta X \cdot \Sigma A_{ij}\delta y_i\delta y_j = 2\delta X \cdot \delta^2e,$$

$$(11c) \quad (\delta X)^2 \cdot \Sigma A_{ij}\delta y_i\delta y_j = \frac{(\delta X)^2}{X} \delta^2e.$$

The fourth expression in (10) is in $\delta X\delta y_j$ (respectively $\delta X\delta y_i$). No terms in this product occur in δ^3E , δ^4E , \cdots , since these terms are at least cubic. So the fourth expression is equal to the terms in this product in the second member of (9). But such terms are absent; thus

$$(11d) \quad X\delta X \cdot \Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) = 0.$$

It follows that $\Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) = 0$, and hence that the fifth expression vanishes,

$$(11e) \quad (\delta X)^2 \cdot \Sigma A_{ij}(y_i\delta y_j + y_j\delta y_i) = 0.$$

The sixth expression in (10) is in $(\delta X)^2$. No terms in this square occur in δ^3E , δ^4E , \cdots , since these terms are at least cubic. So the sixth expression is equal to the terms in $(\delta X)^2$ in the second member of (9). But such terms are absent, so that

$$(11f) \quad (\delta X)^2 \cdot \Sigma A_{ij}y_iy_j = 0.$$

On substituting these six results, (10) becomes

$$\delta^2E = [X + 2\delta X + (\delta X)^2/X]\delta^2e.$$

I.e., the expressions δ^2e and δ^2E are connected by the relation

$$(12) \quad \delta^2E = \frac{(X + \delta X)^2}{X} \delta^2e.$$

Since the coefficient of δ^2e in (12) is positive, it follows that the criterion of stability $\delta^2E(x_1, x_2, x_3, x_4) \geq 0$ and the criterion $\delta^2e(y_1, y_2, y_3) \geq 0$ are equivalent. In both forms of the criterion the sign of equality holds when the variations of the variables satisfy the conditions (3). When these conditions are satisfied, e is constant and the equation

$$\delta E = (X + \delta X)\delta^1e + e\delta X + \frac{(X + \delta X)^2}{X} \delta^2e + \dots$$

reduces to $\delta E = e\delta X$, which represents an obvious physical fact. Since the foregoing method of obtaining the results (8) and (12) is tedious, I should be glad to learn of a simpler way of finding them.

III. GRAPHICAL CONSTRUCTIONS FOR IMAGINARY INTERSECTIONS OF LINE AND CONIC.¹

By R. M. MATHEWS, University of Minnesota.

Introduction.—Several methods commonly known for the graphical solution of a quadratic equation are incomplete, as they are explained only for real roots. The object of this paper is to complete these constructions and to generalize them to find graphically the intersections of an arbitrary line with any conic.

Intersection of a circle and a line.—When a line l cuts a circle O the length of the chord formed is $2\sqrt{r^2 - d^2}$, where r is the radius of the circle and d is the distance from the center O to the chord.

When the line does not cut the circle in real points, $d > r$, the sign of $r^2 - d^2$ is negative and the length of the chord is imaginary. Let us replace the given circle by a second which cuts the line in real points such that the length of the chord formed is $2\sqrt{d^2 - r^2}$. Draw OM perpendicular to the line to cut the circle at T and l at M , and extend OM to O' so

that $MO' = r$ (Fig. 1); with O' as center draw a circle tangent to the given one at T . The length of the chord which it cuts on l is $2\sqrt{d^2 - r^2} = 2i\sqrt{r^2 - d^2}$.

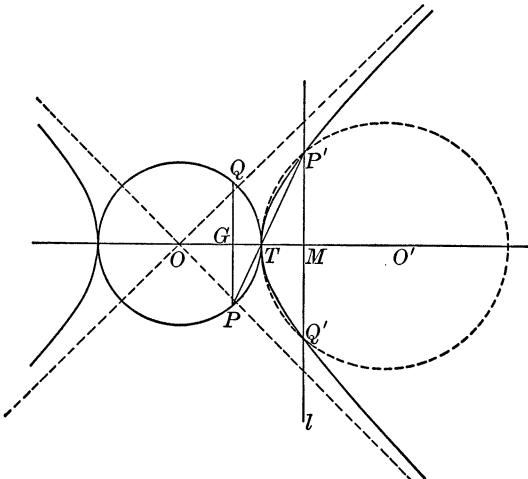


FIG. 1.

¹ Read before the Minnesota Section of the Mathematical Association of America, May 31, 1919.